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Augmenting chains in graphs without a skew star

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Abstract

The augmenting chain technique has been applied to solve the maximum stable set problem in the class of line graphs (which coincides with the maximum matching problem) and then has been extended to the class of claw-free graphs. In the present paper, we propose a further generalization of this approach. Specifically, we show how to find an augmenting chain in graphs containing no skew star, i.e. a tree with exactly three vertices of degree 1 of distances 1, 2, 3 from the only vertex of degree 3. As a corollary, we prove that the maximum stable set problem is polynomially solvable in a class that strictly contains claw-free graphs, improving several existing results.

Keywords: Stable set; Augmenting chain; Polynomial algorithm

1. Introduction

We consider simple undirected graphs without loops and multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The neighborhood of a vertex $v \in V(G)$, denoted $N(v)$, is the subset of vertices of G adjacent to v, and the degree of v is $|N(v)|$. By $S_{i,j,k}$ we denote a tree with exactly three vertices of degree 1 of distances i, j, k from the only vertex of degree 3. In particular, $S_{1,1,1}$ is a *claw*, $S_{1,1,2}$ is a *chair* (called also a *fork*), and $S_{1,2,3}$ is a *skew star* (see Fig. 1). As usual, $K_{1,n}$ denotes the complete bipartite graph with parts of size 1 and n.

A *matching* in a graph is a subset of edges no two of which have a vertex in common, and a *stable set* is a subset of pairwise non-adjacent vertices. The problem of finding a matching of maximum cardinality is a special case of the maximum stable set problem, when restricted

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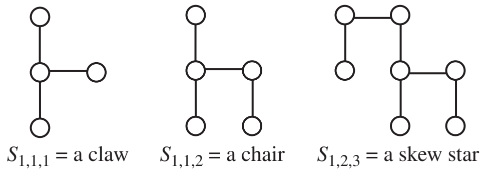


Fig. 1. Examples of $S_{i,j,k}$ graphs.

to the class of line graphs. In general, the maximum stable set problem is NP-hard, while the maximum matching problem is polynomially solvable. The first polynomial-time algorithm to find a maximum matching has been proposed by Edmonds [5]. The algorithm exploits the idea of Berge that a matching M in a graph is maximum if and only if there are no augmenting (alternating) chains for M [3].

Let G be a graph and S a stable set in G. We call the vertices of S *black* and the remaining vertices of the graph *white*. A bipartite graph $H = (W, B, E)$ with parts W and B is called augmenting for S if $|W| > |B|$, $B \subseteq S$, $W \subseteq V(G) - S$, and $N(w) \cap S \subseteq B$ for each vertex $w \in W$. Clearly, if H is augmenting for S, then S is not of maximum cardinality, since $S' = (S - B) \cup W$ is a larger stable set. The converse also is true: if S is not a maximum stable set, and S' is a stable set with $|S'| > |S|$, then the subgraph of G induced by the set $(S - S') \cup (S' - S)$ is augmenting for S. Thus, the problem of finding a stable set of maximum cardinality is polynomially equivalent to detecting augmenting graphs. In general, this is an NP-hard problem. However, if for a certain class of graphs, we have

(a) a complete list of augmenting graphs,
(b) a polynomial-time algorithm for detecting each augmenting graph in the list,

then the maximum stable set problem can be solved efficiently with this approach.

For instance, for the class of claw-free graphs, question (a) has a simple answer. Indeed, by definition, augmenting graphs are bipartite, and each vertex in a claw-free bipartite graph clearly has degree at most 2. Hence, every connected claw-free bipartite graph is either an even cycle or a chain. Cycles of even length and chains of odd length cannot be augmenting, since they have equal number of vertices in both parts. Thus, every connected claw-free augmenting graph is a chain of even length. However, finding augmenting chains is not a trivial task. In 1980, independently, Minty [9] and Sbihi [11] proposed polynomial-time algorithms to determine whether a claw-free graph contains an augmenting chain (thus answering question (b)). Minty's approach reduces this problem to finding a maximum matching in an auxiliary graph. Another way to reduce the maximum stable set problem from claw-free graphs to line graphs has been proposed by Lovász and Plummer in [7]. In 1999, Alekseev [1] extended the solution for claw-free graphs and some other polynomial-time results [4,6] to the class of chair-free graphs. He has shown that every connected chair-free augmenting graph is either a chain or an almost complete bipartite graph (i.e. a graph in which every vertex has at most one non-neighbor in the opposite part), and has proven that both types of augmenting graphs can be found in polynomial time in chair-free graphs.

In the present paper, we extend Minty's approach for finding augmenting chains from claw-free graphs to the class of graphs containing no skew star. More precisely, we prove that augmenting chains in $S_{1,2,3}$-free graphs can be detected in polynomial time. Definitions and notations are given in the next section. Minty's algorithm for detecting augmenting chains in claw-free graphs is outlined in Section 3, while Section 4 is devoted to its extension to $S_{1,2,3}$-free graphs. All proofs are given in Section 4 except the key theorem which is proven in Section 5. As an application, we show in Section 6 that the obtained result leads to a polynomial-time algorithm to find a maximum cardinality stable set in $(S_{1,1,3}, K_{1,n})$-free graphs, generalizing claw-free graphs and some other classes with polynomial-time solvable stable set problem.

2. Preliminaries

Let G be a graph and S be a maximal stable set in G. To determine whether S admits an augmenting chain, we consider two white non-adjacent vertices β and γ, each of which has exactly one black neighbor, respectively, $\overline{\beta}$ and $\overline{\gamma}$. We assume that $\overline{\beta} \neq \overline{\gamma}$ (otherwise the problem is trivial) and any other white vertex is not adjacent to β and γ, and has exactly two black neighbors (the vertices not satisfying the assumption are out of interest, since they cannot occur in any augmenting chain connecting β to γ).

Two white vertices having the same black neighbors will be called *similar*. The similarity is an equivalence relation, and we shall denote the similarity class containing a white vertex v by $c(v)$. Clearly, any augmenting chain contains at most one vertex in each class of similarity. Following Minty's terminology, the similarity classes in the neighborhood of a black vertex b will be called the *wings* of b. Let b be a black vertex different from $\overline{\beta}$ and $\overline{\gamma}$: if b has more than two wings, then b is defined as *regular*, otherwise it is *irregular*. In what follows, R denotes the set of black vertices that are either regular or equal to $\overline{\beta}$ or $\overline{\gamma}$.

An *alternating chain* is a sequence $(x_0, x_1, \ldots, x_k)$ of distinct vertices in which the vertices are alternately white and black. Vertices x_0 and x_k are called the *termini* of the chain. An edge linking two white vertices x_i and x_j with $i \leqslant j-2$ is called a *short chord* if $i = j-2$, and a *long chord* otherwise. If x_0 and x_k are black (respectively, white) vertices, then the sequence is called a black (respectively, white) alternating chain.

Let b_1 and b_2 be two distinct black vertices in R. A black alternating chain with termini b_1 and b_2 is called an IBAP (for irregular black alternating path) if it has no short chord and if all black vertices of the chain, except b_1 and b_2, are irregular. An IWAP (for irregular white alternating path) is a white alternating chain obtained by removing the termini of an IBAP.

An augmenting chain can be represented in different ways. For example, it is a sequence $(I_0 = (\beta), b_0 = \overline{\beta}, I_1, b_1, I_2, \ldots, b_{k-1}, I_{k-1}, b_k = \overline{\gamma}, I_k = (\gamma))$ such that

(a) the b_i $(0 < i < k)$ are distinct black regular vertices,
(b) the I_i $(0 < i < k)$ are pairwise mutually disjoint IWAPs,
(e) each b_i is adjacent to the final terminus of I_i and to the initial one of I_{i+1},
(d) the white vertices in $I_1 \cup \cdots \cup I_{k-1}$ are pairwise non-adjacent.

3. Minty's algorithm

In order to determine whether there exists an augmenting chain with termini β and γ in a claw-free graph, it is sufficient to detect alternating chains with termini β and γ and without short chords. This is a direct corollary of the following simple but important observation.

Observation 1. *An alternating chain* $(\beta = x_0, x_1, \ldots, x_k = \gamma)$ *in a claw-free graph cannot contain a long chord.*

Minty's main idea for detecting augmenting chains in claw-free graphs was to decompose the neighborhood of each black vertex b into at most two subsets $N_1(b)$ and $N_2(b)$, called *node classes*, in such a way that no two vertices in the same node class can occur in the same augmenting chain for S. For vertices $\overline{\beta}$ and $\overline{\gamma}$, such a decomposition is obvious: one of the node classes contains the vertex β (respectively, γ) and the other class includes all the remaining vertices in the neighborhood of $\overline{\beta}$ (respectively, $\overline{\gamma}$). We assume that $N_1(\overline{\beta}) = \{\beta\}$ and $N_1(\overline{\gamma}) = \{\gamma\}$. For an irregular black vertex b, the decomposition also is trivial: the node classes correspond to the wings of b.

Now let b be a regular black vertex. Two white neighbors of b can occur in the same augmenting chain for S only if they are non-similar and non-adjacent. Define an auxiliary graph $H(b)$ as follows: the vertex set of $H(b)$ is $N(b)$ and two vertices u and v of $H(b)$ are linked by an edge if and only if u and v are non-similar non-adjacent vertices in G. The following theorem is a reformulation of Theorem 1 in [9].

Theorem 2. *Let b be any regular black vertex in a claw-free graph. Then*

(a) *$H(b)$ is bipartite, and*
(b) *two non-similar neighbors of b are non-adjacent in G if and only if they belong to different parts of $H(b)$.*

The two node classes $N_1(b)$ and $N_2(b)$ of a regular black vertex b, therefore, correspond to the two parts of the bipartite graph $H(b)$.

Minty has shown how to determine the pairs (u, v) of vertices such that there exists an IWAP with termini u and v. More precisely, let b_0 be a black vertex in R, and let W_1 be one of its wings ($W_1 = N_2(\overline{\beta})$ if $b_0 = \overline{\beta}$, and $W_1 = N_2(\overline{\gamma})$ if $b_0 = \overline{\gamma}$). The set P of pairs (u, v) such that u belongs to W_1 and is a terminus of an IWAP can be determined in polynomial time as follows:

1. Set $k := 1$.
2. Let b_k denote the second black neighbor of the vertices in W_k; if b_k has two wings then go to Step 3. If b_k is regular and different from b_0 then go to Step 4. Otherwise STOP: P is empty.
3. Let W_{k+1} denote the second wing of b_k. Set $k := k + 1$ and go to Step 2.
4. Construct an auxiliary graph with vertex set $W_1 \cup \cdots \cup W_k$ and link two vertices by an edge if and only if they are non-adjacent in G and belong to two consecutive sets W_i and W_{i+1}. Orient all edges from W_i to W_{i+1}.
5. Determine the set P of pairs (u, v) such that $u \in W_1$, $v \in W_k$ and there exists a path from u to v in the auxiliary graph.

The last important concept used in Minty's algorithm is *Edmonds' Graph* which is constructed as follows:

- For each black vertex $b \in R$ do the following: create two vertices b_1 and b_2, link them by a black edge, and identify b_1 and b_2 with the two node classes $N_1(b)$ and $N_2(b)$ of b. In particular, $\overline{\beta}_1$ represents $N_1(\overline{\beta}) = \{\beta\}$ and $\overline{\gamma}_1$ represents $N_1(\overline{\gamma}) = \{\gamma\}$.
- Create two vertices β and γ, and link β to $\overline{\beta}_1$ and γ to $\overline{\gamma}_1$ by a white edge.
- Link b_i (i=1 or 2) to b'_j (j=1 or 2) with a white edge if there are two white vertices u and v in G such that $u \in N_i(b)$, $v \in N_j(b')$, and there exists an IWAP with termini u and v. Identify each such white edge with a corresponding IWAP.

The black edges define a matching in the Edmonds' graph. If the matching is not maximum, then there exists an augmenting chain of edges $(e_0, \ldots, e_{2k})$ such that the even indexed edges are white, the odd-indexed edges are black, e_0 is the edge linking β to $\overline{\beta}_1$, and e_{2k} is the edge linking γ to $\overline{\gamma}_1$. Such an augmenting chain of edges in the Edmonds' graph corresponds to an alternating chain C in G. Indeed, notice first that each white edge e_i with $2 \leqslant i \leqslant 2k-2$ corresponds to an IWAP whose termini will be denoted w_{i-1} and w_i. Also, each black edge e_i with $1 \leqslant i \leqslant 2k-1$ corresponds to a black vertex b_i. The alternating chain C is obtained as follows:

- Replace e_0 by β, e_{2k} by γ, and each white edge $e_i (2 \leqslant i \leqslant 2k-2)$ by an IWAP with termini w_{i-1} and w_i.
- Replace each black edge $e_i (1 \leqslant i \leqslant 2k-1)$ by the vertex b_i.

This alternating chain C in G has no short chord. Indeed, IWAPs have no short chord, and it follows from the second part of Theorem 2 that there is no edge linking w_{i-1} and w_i for an odd i since w_{i-1} and w_i are two non-similar vertices that belong to different node classes of b_{i-1}. Hence, as observed at the beginning of this section, C is an augmenting chain. Conversely, as observed in [9], an augmenting chain C in G corresponds to an augmenting chain of edges in the Edmonds' graph. In other words, determining whether there exists an augmenting chain in G with termini β and γ is equivalent to determining whether there exists an augmenting chain of edges in the Edmonds' graph. The latter problem is polynomially solvable [5]. Minty's algorithm can now be summarized as follows.

Minty's algorithm for finding augmenting chains in claw free graphs:

1. Partition the neighborhood of each regular black vertex b into two node classes $N_1(b)$ and $N_2(b)$ by constructing the bipartite graph $H(b)$ in which two white neighbors of b are linked by an edge if and only if they are non-adjacent and non-similar.
2. Determine the set of pairs (u, v) of (not necessarily distinct) white vertices such that there exists an IWAP with termini u and v.
3. Construct the Edmonds' graph.
4. If the Edmonds' graph contains an augmenting chain of edges, then it corresponds to an augmenting chain in G with termini β and γ; otherwise, there are no augmenting chains with termini β and γ.

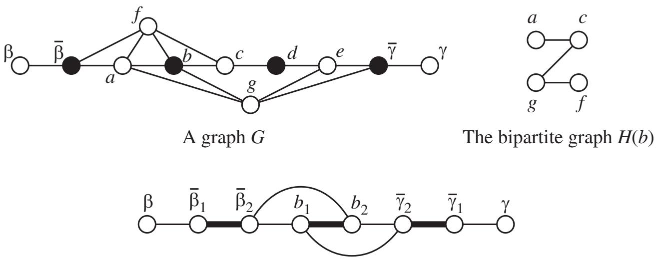
All concepts defined in this section are illustrated in Fig. 2. The graph G has one regular black vertex (vertex b) and one irregular black vertex (vertex d). The bipartite graph $H(b)$ defines the partition of $N(b)$ into two node classes $N_1(b) = \{a, g\}$ and $N_2(b) = \{c, f\}$. The corresponding Edmonds' graph is represented with bold lines for the black edges and regular lines for the white edges. There are four IWAPs: (a), (f), (g) and (c, d, e) represented, respectively, by the white edges $\overline{u}_2 b_1$, $\overline{u}_2 b_2$, $b_1 \overline{v}_2$ and $b_2 \overline{v}_2$. The Edmonds' graph contains two augmenting chains: $(u, \overline{u}_1, \overline{u}_2, b_1, b_2, \overline{v}_2, \overline{v}_1, v)$ and $(u, \overline{u}_1, \overline{u}_2, b_2, b_1, \overline{v}_2, \overline{v}_1, v)$ which correspond to the augmenting chains $(u, \overline{u}, a, b, c, d, e, \overline{v}, v)$ and $(u, \overline{u}, f, b, g, \overline{v}, v)$ in G.

4. Extension to graphs without skew star

We first show that, like for claw-free graphs (see Observation 1), in order to determine whether there exists an augmenting chain in an $S_{1,2,3}$-free graph, it is sufficient to detect alternating chains with termini β and γ and without short chords.

Lemma 3. *An alternating chain $(\beta = x_0, x_1, \ldots, x_k = \gamma)$ in an $S_{1,2,3}$-free graph cannot contain a long chord.*

Proof. Assume that there is a long chord $x_i x_j$ with $j > i + 2$. Without loss of generality, we can assume that $j - i$ is maximum. Let r be the smallest index and s the largest one such that



A graph G

The bipartite graph $H(b)$

The corresponding Edmond's graph

Fig. 2. Illustration of Minty's algorithm.

$x_i x_r \in E(G)$ and $x_j x_s \in E(G)$. But now the vertices $x_{r-1}, x_r, x_i, x_j, x_s, x_{s+1}, x_{i+1}$ induce a skew star in G, a contradiction. □

The partition of $N(\overline{\beta})$ and $N(\overline{\gamma})$ into two node classes is done as in Minty's algorithm. However, this partition has to be modified for a regular black vertex b. Indeed, notice that the graph $H(b)$ defined in the previous section is not necessarily bipartite when considering $S_{1,2,3}$-free graphs. For example, assume that b has three pairwise non-similar non-adjacent neighbors. Then these three vertices are pairwise adjacent in $H(b)$ which means that $H(b)$ is not bipartite. We propose to modify the definition of $H(b)$ by imposing an additional condition for creating an edge in $H(b)$. But before that, let us prove some useful lemmas.

Lemma 4. *Let G be an $S_{1,2,3}$-free graph, and let A ($\beta = x_0, x_1, \ldots, x_k = \gamma$) be an alternating chain in G. Then every vertex in G has at most two pairwise non-adjacent neighbors on A.*

Proof. Let v be a vertex having at least three non-adjacent neighbors x_r, x_s and x_t on A, with $r < s < t$. We know from Lemma 3 that A has no long chords, which means that v is not on A. We can assume that r is minimum and t is maximum. Hence, vertices $x_{r-2}, x_{r-1}, x_r, v, x_t, x_{t+1}, x_s$ (if $x_r x_{r-2} \notin E(G)$) or $x_{r-3}, x_{r-2}, x_r, v, x_t, x_{t+1}, x_s$ (if $x_r x_{r-2} \in E(G)$) induce a skew star in G, a contradiction. □

Lemma 5. *Let G be an $S_{1,2,3}$-free graph, let A ($\beta = x_0, x_1, \ldots, x_k = \gamma$) be an alternating chain in G, and let x_i be any black regular vertex on A. If x_i has three pairwise non-similar neighbors u, v and w with $c(u) = c(x_{i-1})$ and $c(v) = c(x_{i+1})$, then G contains an odd number of edges among uv, uw and vw.*

Proof. *Case* 1: assume that u, v and w are pairwise non-adjacent. We know from Lemma 4 that w cannot have a neighbor both on $(x_0, \ldots, x_{i-2})$ and on $(x_{i+2}, \ldots, x_k)$. Without loss of generality suppose w has no neighbor on $(x_0, \ldots, x_{i-2})$. But then either vertices $x_{i-3}, x_{i-2}, u, x_i, v, x_{i+2}, w$ (if $ux_{i-3} \notin E(G)$) or $x_{i-4}, x_{i-3}, u, x_i, v, x_{i+2}, w$ (if $ux_{i-3} \in E(G)$) induce a skew star in G, a contradiction.

Case 2: assume that $uv \notin E(G)$, $uw \in E(G)$ and $vw \in E(G)$. Since w is neither similar to u, nor to v, we know that the second black neighbor $\overline{w}$ of w is different from x_{i-2} and x_{i+2}. According to Lemma 4, we know that $\overline{w}$ is not on the chain ($\beta = x_0, x_1, \ldots, x_{i-2}, u, x_i, v, x_{i+2}, \ldots, x_k = \gamma$). Let r be the smallest index and s the largest one such that w is adjacent to x_r and x_s (possibly

$x_r = u$ and/or $x_s = v$). Then either vertices $x_{r-2}, x_{r-1}, x_r, w, x_s, x_{s+1}, \overline{w}$ (if $x_r x_{r-2} \notin E(G)$) or $x_{r-3}, x_{r-2}, x_r, w, x_s, x_{s+1}, \overline{w}$ (if $x_r x_{r-2} \in E(G)$) induce a skew star in G, a contradiction.

Case 3: assume that $vw \notin E(G)$, $uv \in E(G)$ and $uw \in E(G)$ (the case where $uw \notin E(G)$, $uv \in E(G)$ and $vw \in E(G)$ is symmetrical). If w has no neighbor on $(x_{i+3}, \ldots, x_k)$, then either vertices $x_{i+3}, x_{i+2}, v, u, w, \overline{w}, x_{i-2}$ (if $vx_{i+3} \notin E(G)$) or $x_{i+4}, x_{i+3}, v, u, w, \overline{w}, x_{i-2}$ (if $vx_{i+3} \in E(G)$) induce a skew star in G, a contradiction. So w has at least one neighbor on $(x_{i+3}, \ldots, x_k)$, and we know from Lemma 4 that it has no neighbor on $(x_0, \ldots, x_{i-2})$. Let j be the largest index such that $wx_j \in E(G)$. If $j = i+3$, then either vertices $x_{i-2}, u, w, x_{i+3}, x_{i+4}, x_{i+5}, x_{i+2}$ (if $x_{i+3}x_{i+5} \notin E(G)$) or $x_{i-2}, u, w, x_{i+3}, x_{i+5}, x_{i+6}, x_{i+2}$ (if $x_{i+3}x_{i+5} \in E(G)$) induce a skew star in G, a contradiction. If $j > i+3$ then either vertices $x_{j+1}, x_j, w, u, x_{i-2}, x_{i-3}, v$ (if $ux_{i-3} \notin E(G)$) or $x_{j+1}, x_j, w, u, x_{i-3}, x_{i-4}, v$ (if $ux_{i-3} \in E(G)$) induce a skew star in G, a contradiction. □

Definition. A pair (u, v) of vertices is *special* if u and v have a common black regular neighbor b, and if there is a vertex $w \in N(b)$ which is similar neither to u nor to v and such that either both of uw and vw or none of them is an edge in G.

Lemma 6. *If (u, v) is a special pair of non-adjacent non-similar vertices, then u and v cannot occur in the same augmenting chain.*

Proof. Let (u, v) be a special pair of non-adjacent non-similar vertices, and let b be the common black regular neighbor of u and v. If an augmenting chain $(\beta = x_0, x_1, \ldots, x_k = \gamma)$ contains both u and v, then clearly $u = x_{i-1}$, $b = x_i$ and $v = x_{i+1}$ for some odd index i. Since u and v are non-adjacent, it follows from Lemma 5 that each vertex in $N(b)$ that is similar neither to u nor to v has exactly one neighbor in $\{u, v\}$. Hence, the pair (u, v) is not special, a contradiction. □

For a regular black vertex b, we therefore, define the graph $H(b)$ as follows: the vertex set of $H(b)$ is $N(b)$, and two vertices u and v in $H(b)$ are linked by an edge if and only if (u, v) is a pair of non-special non-similar non-adjacent vertices in G.

Notice that claw-free graphs do not contain pairs of special non-adjacent non-similar vertices. Indeed if such a pair (u, v) exists, then let b and w be vertices as in the above definition, and let $\overline{w}$ be the second black neighbor of w. Then vertices b, u, v and w or vertices u, v, w and $\overline{w}$ induce a claw. Hence, the modified graph $H(b)$ coincides with the original one in the case of claw-free graphs. When extended to the class of $S_{1,2,3}$-free graphs, the modification can differ from the original definition. However, the important thing is that the new graph $H(b)$ remains bipartite whenever we deal with $S_{1,2,3}$-free graphs.

Theorem 7. *Let b be any regular black vertex in an $S_{1,2,3}$-free graph. Then $H(b)$ is bipartite.*

This key theorem will be proven in the next section. In the present one, we shall use the result of Theorem 7 to complete the construction of a polynomial-time algorithm for finding augmenting chains in graphs without a skew star.

As in the previous section, we define the two node classes $N_1(b)$ and $N_2(b)$ of a regular black vertex b to be the two parts of the bipartite graph $H(b)$. Notice, however, that the partition of $H(b)$ into two node classes is not unique when $H(b)$ has more than one connected component. More importantly, the second part of Theorem 2 is not valid for $S_{1,2,3}$-free graphs, since it may happen that two non-similar vertices belonging to different node classes of a regular black vertex

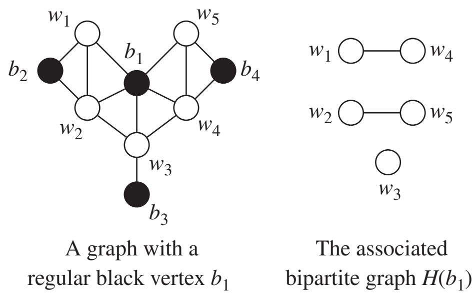


Fig. 3. Non-similar adjacent vertices can belong to different node classes.

b are adjacent in G. For illustration, consider the graph G depicted on the left-hand side of Fig. 3. It contains a regular black vertex b_1 with three wings $\{w_1, w_2\}$, $\{w_3\}$ and $\{w_4, w_5\}$. The corresponding bipartite graph $H(b_1)$ is depicted on the right-hand side of Fig. 3. If we place w_3 in the same part of $H(b_1)$ as w_1, then w_3 and w_4 constitute a pair of non-similar vertices that belong to different node classes and are adjacent in G.

Clearly, an isolated vertex in $H(b)$ cannot belong to any augmenting chain. Hence, an IWAP in an augmenting chain necessarily connects two white vertices that are not isolated in the respective bipartite graphs associated with their black neighbors in R. This motivates the following definition.

Definition. Let $(b_1, w_1, \ldots, w_{k-1}, b_k)$ be an IBAP. The IWAP obtained by removing b_1 and b_k is *interesting* if w_1 and w_{k-1} are non-isolated vertices in $H(b_1)$ and $H(b_k)$, respectively.

Let W denote the set of white vertices w which have a black neighbor $b \in R$ such that w is an isolated vertex in $H(b)$. The set of pairs (u, v) such that there is an interesting IWAP with termini u and v can be determined in polynomial time by using the algorithm of the previous section, and by removing a pair (u, v) if u or/and v belongs to W.

Augmenting chains in graphs without skew star are detected by using Edmonds' graph, which is constructed as in Minty's algorithm, except that white edges in the Edmonds' graph correspond to interesting IWAPs. As shown in the previous section, an augmenting chain of edges in the Edmonds' graph corresponds to an alternating chain C in G. It follows from Lemma 3 that in order to prove that C is augmenting, it is sufficient to prove that C has no short chord. Since IWAPs have no short chords, it remains to prove that given any regular black vertex on C, its two white neighbors on C are non-adjacent. We first prove a useful lemma.

Lemma 8. *Let b be a regular black vertex and let v_1, v_2, v_3 and v_4 be four vertices in $N(b)$. If $H(b)$ contains the edges v_1v_2 and v_3v_4 but does not contain the edges v_1v_3, v_1v_4 and v_2v_4, then v_1, v_2, v_3, v_4 belong to at most three different similarity classes.*

Proof. By contradiction, assume that v_1, v_2, v_3 and v_4 belong to four different similarity classes. Let $\overline{v}_1, \overline{v}_2, \overline{v}_3$ and $\overline{v}_4$ denote their second black neighbors. Since $H(b)$ contains the edges v_1v_2 and v_3v_4, we know that $v_1v_2 \notin E(G)$ and $v_3v_4 \notin E(G)$, and that the pair (v_1, v_2) is non-special. Hence, $E(G)$ contains exactly one of the edges v_1v_3 and v_2v_3. We can assume that $v_1v_3 \in E(G)$ and $v_2v_3 \notin E(G)$. Indeed, this is the case if v_2 is adjacent to v_3 in $H(b)$; otherwise, v_1 and v_3 play a symmetric role and the assumption is therefore valid without loss of generality.

Now, since (v_1, v_2) and (v_3, v_4) are non-special pairs of vertices, we have $v_1v_4 \notin E(G)$ and $v_2v_4 \in E(G)$. Vertices v_1 and v_4 are non-similar and non-adjacent vertices in G while they are not adjacent in $H(b)$. This means that the pair (v_1, v_4) is special, and there is therefore a vertex $u \in N(b)$ which is similar neither to v_1, nor to v_4, and such that either both of uv_1 and uv_4 or none of them belong to $E(G)$. Since $c(v_2) \neq c(v_3)$, we can assume by symmetry that $c(u) \neq c(v_2)$.

If both uv_1 and uv_4 belong to $E(G)$, then $uv_2 \notin E(G)$ (since (v_1, v_2) is non-special), and vertices $\overline{v}_1, v_1, u, v_4, v_2, \overline{v}_2, \overline{v}_4$ induce a skew star in G, a contradiction. If $uv_1 \notin E(G)$ and $uv_4 \notin E(G)$, then $uv_2 \in E(G)$ (since (v_1, v_2) is non-special) and $uv_3 \notin E(G)$, else vertices $v_1, v_3, u, v_2, v_4, \overline{v}_4, \overline{v}_2$ induce a skew star in G. It follows that $c(u) = c(v_3)$ (since (v_3, v_4) is non-special), and vertices $v_3, \overline{v}_3, u, v_2, v_4, \overline{v}_4, \overline{v}_2$ induce a skew star in G, a contradiction. □

Lemma 9. *Let $A(\beta = x_0, x_1, \ldots, x_k = \gamma)$ be an alternating chain in G such that*

- $x_{i-1}x_{i+1} \notin E(G)$ if x_i *is an irregular black vertex,*
- x_{i-1} *and* x_{i+1} *are non-isolated vertices in* $H(x_i)$ if x_i *is a regular black vertex.*

Then A has no short chord.

Proof. Assume there is a short chord $x_{i-1}x_{i+1} \in E(G)$ for some regular black vertex x_i. Assume that $x_{i-1} \in N_1(x_i)$ and $x_{i+1} \in N_2(x_i)$. Since x_{i-1} and x_{i+1} are non-isolated vertices in $H(x_i)$, there exist two vertices u and v such that u is linked to x_{i-1} and v to x_{i+1} in $H(x_i)$. According to Lemma 8, vertices u, v, x_{i-1} and x_{i+1} belong to at most three similarity classes.

Case 1: assume that $c(u) \neq c(x_{i+1})$. Then $ux_{i+1} \notin E(G)$ since (u, x_{i-1}) is non-special. Moreover, since u and x_{i+1} are non-adjacent non-similar vertices while u is not linked to x_{i+1} in $H(x_i)$, there must exist a vertex $w \in N(b)$ non-similar to u and x_{i+1} that makes the pair (u, x_{i+1}) special. Hence, w sees either both or none of u and x_{i+1} in G. We now know that $c(w) \neq c(x_{i-1})$, else the triplet w, x_{i+1}, u contradicts Lemma 5. Moreover, w sees exactly one vertex among u and x_{i-1} in G (since (u, x_{i-1}) is non-special), and this implies that w sees exactly one vertex among x_{i+1} and x_{i-1} in G. There are therefore exactly two edges among wx_{i-1}, wx_{i+1} and $x_{i-1}x_{i+1}$ in G, which contradicts Lemma 5.

Case 2: we can now assume $c(u) = c(x_{i+1})$ and $c(v) = c(x_{i-1})$ (by symmetry). Since x_i is regular, there exists a vertex w non-similar to x_{i-1} and x_{i+1}. Let $\overline{w}$ denote its second black neighbor. According to Lemma 5, w sees either both or none of x_{i-1} and x_{i+1}. Hence, w sees either both or none of u and v, else (u, x_{i-1}) or (v, x_{i+1}) is a special pair. Also, we know that $uv \in E(G)$, else the triplet u, v, w contradicts Lemma 5. In summary, we can assume that w sees both u and v and none of x_{i-1} and x_{i+1} (else w sees both x_{i-1} and x_{i+1} and none of u and v and one can permute the roles of x_{i-1} and x_{i+1} with those of u and v). We can also assume, by symmetry, that $w \in N_1(x_i)$. Since w and x_{i-1} are non-adjacent in $H(x_i)$ while they are non-adjacent and non-similar in G, there must exist a vertex y that makes the pair (w, x_{i-1}) special. Vertex y cannot be similar to x_{i+1}, else the triplet x_{i-1}, y, w contradicts Lemma 5. If y sees both w and x_{i-1} in G, then $yx_{i+1} \in E(G)$ and $yu \notin E(G)$ by Lemma 5, and vertices $x_{i-2}, x_{i-1}, y, w, u, x_{i+2}, \overline{w}$ induce a skew star in G, a contradiction. Hence, y sees none of w and x_{i-1} in G, and we now have $yx_{i+1} \in E(G)$ by Lemma 5. Let A_L and A_R denote the subsequences $(x_0, \ldots, x_{i-3})$ and $(x_{i+3}, \ldots, x_k)$, respectively. If w and y have no neighbor on A_L, then vertices $x_{i-3}, x_{i-2}, x_{i-1}, x_i, w, \overline{w}, y$ (if $x_{i-1}x_{i-3} \notin E(G)$) or $x_{i-4}, x_{i-3}, x_{i-1}, x_i, w, \overline{w}, y$ (if $x_{i-1}x_{i-3} \in E(G)$) induce a skew star in G, a contradiction. Hence, w or y has a neighbor on A_L and, by symmetry, w or y has a neighbor on A_R. But we know from Lemma 4 that neither w nor y can have a neighbor both on A_L and on A_R. Hence, by symmetry, we can assume that w has a neighbor on A_L and no on A_R, while

y has a neighbor on A_R and no on A_L. Let r be the smallest index such that $wx_r \in E(G)$ and let x_s be any neighbor of y on A_R. Then vertices $x_s, y, x_i, w, x_{i-3}, x_{i-4}, \overline{w}$ (if $r = i-3$) or $x_{r-1}, x_r, w, x_i, y, x_s, x_{i-1}$ (if $r < i-3$) induce a skew star in G, a contradiction. □

In summary, the proposed algorithm for finding augmenting chains in graphs without skew star works as follows.

Algorithm for finding augmenting chains in graphs without skew star:

1. Partition the neighborhood of each regular black vertex b into two node classes $N_1(b)$ and $N_2(b)$ by constructing the bipartite graph $H(b)$ in which two white neighbors u and v of b are linked by an edge if and only if (u, v) is a pair of non-special non-adjacent non-similar vertices.
2. Determine the set of pairs (u, v) of (not necessarily distinct) white vertices such that there exists an interesting IWAP with termini u and v.
3. Construct the Edmond's graph.
4. If the Edmond's graph contains an augmenting chain of edges, then it corresponds to an augmenting chain in G with termini β and γ; otherwise, there are no augmenting chains with termini β and γ.

The above algorithm is very similar to Minty's algorithm. It only differs in Step 1 where an additional condition is imposed for introducing an edge in $H(b)$, and in Step 2 where only interesting IWAPs are considered.

5. Graph $H(b)$ is bipartite

In this section we prove Theorem 7 that states that $H(b)$ is bipartite for every regular black vertex b. To simplify the notations, we use H instead of $H(b)$. If H is not bipartite, then it contains an induced odd chordless cycle. We first show that the vertices on such an odd cycle belong to exactly three similarity classes.

Lemma 10. *If H is not bipartite, then the vertices on any induced odd chordless cycle in H belong to exactly three similarity classes.*

Proof. Assume H is not bipartite, and let $C(v_0, v_1, \ldots, v_k, v_0)$ be an induced odd chordless cycle in H. In what follows, all indices in C will be taken modulo $k+1$. Since C has an odd length and adjacent vertices in H belong to different similarity classes, we know that the vertices on C belong to at least three similarity classes.

It remains to show that at most three similarity classes can appear on C. This is clearly the case if $k = 2$. If $k > 2$, then it follows from Lemma 8 that C contains at least two similar vertices. Let v_i and v_j be two similar vertices on C such that the chain $P = (v_i, v_{i+1}, \ldots, v_j)$ contains an even number of vertices. We may assume that the pair (v_i, v_j) is minimal in the sense that there is no other such pair on P.

Notice that $c(v_{i+2}) \neq c(v_i)$ and $c(v_{j-1}) \neq c(v_{i+1})$, else (v_{i+2}, v_j) and (v_{i+1}, v_{j-1}) would contradict the minimality of (v_i, v_j). It follows that $c(v_{j-1}) = c(v_{i+2})$, since otherwise vertices $v_{i+1}, v_{i+2}, v_{j-1}$, and v_j would contradict Lemma 8.

Suppose now there exists a vertex v_r on C such that $c(v_r) \notin \{c(v_i), c(v_{i+1}), c(v_{i+2})\}$. Then, one of the three sets $\{v_r, v_{r+1}, v_i, v_{i+1}\}$, $\{v_r, v_{r+1}, v_{i+1}, v_{i+2}\}$, $\{v_r, v_{r+1}, v_{j-1}, v_j\}$ contradicts Lemma 8. □

Consider now any cyclic sequence $\mathcal{S} = (v_0, v_1, \dots, v_k, v_0)$ of vertices in $V(H)$. We will say that $\mathcal{S}$ has property $\mathcal{P}$ if the three following conditions are satisfied:

- $k \geqslant 2$ is even,
- consecutive vertices on $\mathcal{S}$ are non-similar,
- the vertices in $\mathcal{S}$ belong to exactly three similarity classes.

Up to this point, we have shown that if H is not bipartite, then the vertices on any induced odd chordless cycle in H define a cyclic sequence $\mathcal{S}$ with the property $\mathcal{P}$. In the rest of this section, the indices in $\mathcal{S}$ will be taken modulo $k+1$. With each cyclic sequence $\mathcal{S}$ we associate a graph, denoted $G_{\mathcal{S}}$, built as follows:

- For each pair (v_i, v_j) of non-similar vertices in $\mathcal{S}$, we create a vertex in $G_{\mathcal{S}}$.
- For each triplet (v_i, v_{i+1}, v_j) of pairwise non-similar vertices, we create an edge in $G_{\mathcal{S}}$ linking vertex (v_i, v_j) to vertex (v_{i+1}, v_j).

Lemma 11. *If H is not bipartite, then the vertices on any induced odd chordless cycle in H define a cyclic sequence $\mathcal{S}$ for which $G_{\mathcal{S}}$ is bipartite.*

Proof. Assume H is not bipartite, and let $\mathcal{S} = (v_0, v_1, \dots, v_k, v_0)$ be the cyclic sequence of vertices of an induced odd chordless cycle in H. Given any two adjacent vertices (v_i, v_j) and (v_{i+1}, v_j) in $G_{\mathcal{S}}$, we know that exactly one among v_iv_j and $v_{i+1}v_j$ belongs to $E(G)$. Indeed, if this is not the case, then (v_i, v_{i+1}) is a special pair which contradicts the fact that $v_iv_{i+1} \in E(H)$. It follows that for any chain with an odd number of vertices linking vertex (v_i, v_j) to vertex (v_r, v_s) in $G_{\mathcal{S}}$, either both of v_iv_j and v_rv_s or none of them belong to $E(G)$.

Now suppose that $G_{\mathcal{S}}$ contains an odd cycle $\mathcal{O}$. Consider two consecutive vertices (v_i, v_j) and (v_{i+1}, v_j) on $\mathcal{O}$. On the one hand, we have shown that exactly one among v_iv_j and $v_{i+1}v_j$ belongs to $E(G)$. On the other hand, there is a chain on $\mathcal{O}$ with an odd number of vertices linking vertex (v_i, v_j) to vertex (v_{i+1}, v_j), which means that either both of v_iv_j and $v_{i+1}v_j$ or none of them belong to $E(G)$, a contradiction. $\square$

An ordered pair (v_p, v_q) of non-similar vertices on $\mathcal{S}$ is said to be *maximal* if the vertices on the subsequence $(v_p, v_{p+1}, \dots, v_q)$ alternatively belong to similarity classes $c(v_p)$ and $c(v_q)$, while v_{p-1} and v_{q+1} belong to the third similarity class, different from $c(v_p)$ and $c(v_q)$. Notice that a cyclic sequence $\mathcal{S}$ with property $\mathcal{P}$ necessarily contains such a maximal ordered pair of non-similar vertices. Given any cyclic sequence $\mathcal{S}$ with property $\mathcal{P}$, the following algorithm builds a new cyclic sequence $\mathcal{S}'$, called *contraction* of $\mathcal{S}$.

Contraction algorithm:

1. Without loss of generality, we may assume that the vertices on $\mathcal{S}$ are labeled so that (v_p, v_0) is a maximal ordered pair of non-similar vertices for some even index p. Set $w_0 := v_0$, $r := 0$ (counter for the vertices on $\mathcal{S}'$) and $s := 0$ (position of w_r on $\mathcal{S}$, i.e. $w_r = v_s$).
2. Determine the smallest even integer $t \geqslant 0$ such that v_s is similar to v_{s+t} but not to v_{s+t+2}.
3. If $s + t = k$, then go to Step 4; else set $w_{r+1} := v_{s+t+1}$, $r := r + 1$, $s := s + t + 1$, and go to Step 2.
4. Set $\mathcal{S}' = (w_0, w_1, \dots, w_r, w_0)$ and STOP.

The graphs on the left-hand side of Fig. 4 correspond to successive contractions of a cyclic sequence. The colors on the vertices correspond to the different similarity classes.

Lemma 12. *Let $\mathcal{S}'$ be the contraction of a cyclic sequence $\mathcal{S}$ with property $\mathcal{P}$. Let w_r be any vertex on $\mathcal{S}'$ and let s be the corresponding index on $\mathcal{S}$, i.e. $w_r = v_s$. Then, v_{s-1}, v_s and v_{s+1} are pairwise non-similar.*

Proof. Since consecutive vertices on $\mathcal{S}$ are non-similar, it is sufficient to prove that v_{s-1} and v_{s+1} are non-similar. The vertices on $\mathcal{S}$ are supposed to be labeled so that (v_p, v_0) is a maximal ordered pair of non-similar vertices for some even index p. Notice first that $c(v_k) \neq c(v_1)$, since no vertex on the chain $(v_p, \dots, v_k, v_0)$ is similar to v_1. Hence, the result holds for $r = 0$. If $r > 0$, then let s be the index such that $w_r = v_s$. It follows from Steps 2 and 3 of the contraction algorithm that $c(v_{s+1}) \neq c(w_{r-1}) = c(v_{s-1})$. □

Lemma 13. *The contraction algorithm is finite.*

Proof. Remember that the vertices on $\mathcal{S}$ are labeled so that (v_p, v_0) is a maximal ordered pair of non-similar vertices for some even index p. Let r and s be two indices such that $w_r = v_s$, and let $t \geqslant 0$ be the smallest even number such that $c(v_{s+t}) = c(v_s)$ and $c(v_{s+t+2}) \neq c(v_s)$. Since s strictly increases at each execution of Step 3, we consider the situation (which will sooner or later occur) where $s \leqslant p$ while $s + t \geqslant p$. Since v_{p+1} is not similar to v_{p-1}, we know that v_s is similar to v_p, else $s + t \leqslant p - 1$. Moreover, $v_p, v_{p+2}, \dots, v_k$ are similar vertices while $c(v_k) \neq c(v_1)$. Hence, $s + t = k$ and the algorithm stops at Step 4. □

Lemma 14. *If $\mathcal{S}$ is a cyclic sequence with property $\mathcal{P}$, then its contraction $\mathcal{S}'$ also is a cyclic sequence with property $\mathcal{P}$.*

Proof. Again, we assume that the vertices on $\mathcal{S}$ are labeled so that (v_p, v_0) is a maximal ordered pair of non-similar vertices for some even index p. Consider two consecutive vertices $w_r = v_s$ and $w_{r+1} = v_{s+t+1}$ on $\mathcal{S}'$. Since $c(v_s) = c(v_{s+t}) \neq c(v_{s+t+1})$ we know that consecutive vertices on $\mathcal{S}'$ are non-similar. Moreover, since neither v_{p-1} nor v_p is similar to $v_0 = w_0$, we have $w_1 = v_s$ with $s \leqslant p - 1$. Hence, $\mathcal{S}'$ has at least two vertices. It remains to prove that $\mathcal{S}'$ contains an odd number of vertices.

In Step 3 of the contraction algorithm, we skip directly from $w_r = v_s$ to $w_{r+1} = v_{s+t+1}$, which means that vertices $v_{s+1}, v_{s+2}, \dots, v_{s+t}$ are no longer considered in $\mathcal{S}'$. Hence, $\mathcal{S}'$ is obtained from $\mathcal{S}$ by removing an even number of vertices which means that $\mathcal{S}'$ contains an odd number of vertices. □

Lemma 15. *Let $\mathcal{S}'$ be the contraction of a cyclic sequence $\mathcal{S}$ with property $\mathcal{P}$. If $\mathcal{S}' = \mathcal{S}$, then $G_{\mathcal{S}}$ is not bipartite.*

Proof. Let $\mathcal{S} = (v_0, v_1, \dots, v_k, v_0)$ be a cyclic sequence with property $\mathcal{P}$, and assume that its contraction $\mathcal{S}' = (w_0, w_1, \dots, w_r, w_0)$ is equal to $\mathcal{S}$. If $c(v_{i+2}) = c(v_i)$ for some index i, then vertices v_{i+1} and v_{i+2} would be removed from $\mathcal{S}$ to obtain $\mathcal{S}'$, a contradiction. Hence, two vertices v_i and v_j in $\mathcal{S}$ are similar if and only if $j = i \bmod 3$. Hence k is equal to $6h + 2$ for some integer $h \geqslant 0$. But this implies that $G_{\mathcal{S}}$ contains the odd cycle on vertices (v_0, v_{3h+2}), (v_{3h+2}, v_1), (v_1, v_{3h+3}), $(v_{3h+3}, v_2), \dots, (v_{3h}, v_k)$, (v_k, v_{3h+1}) and (v_{3h+1}, v_0). □

The graph at the bottom of the left-hand side of Fig. 4 corresponds to a cyclic sequence $\mathcal{S}$ whose contraction $\mathcal{S}'$ is equal to $\mathcal{S}$. To the right of this cyclic sequence, we exhibit an odd cycle in $G_{\mathcal{S}}$.

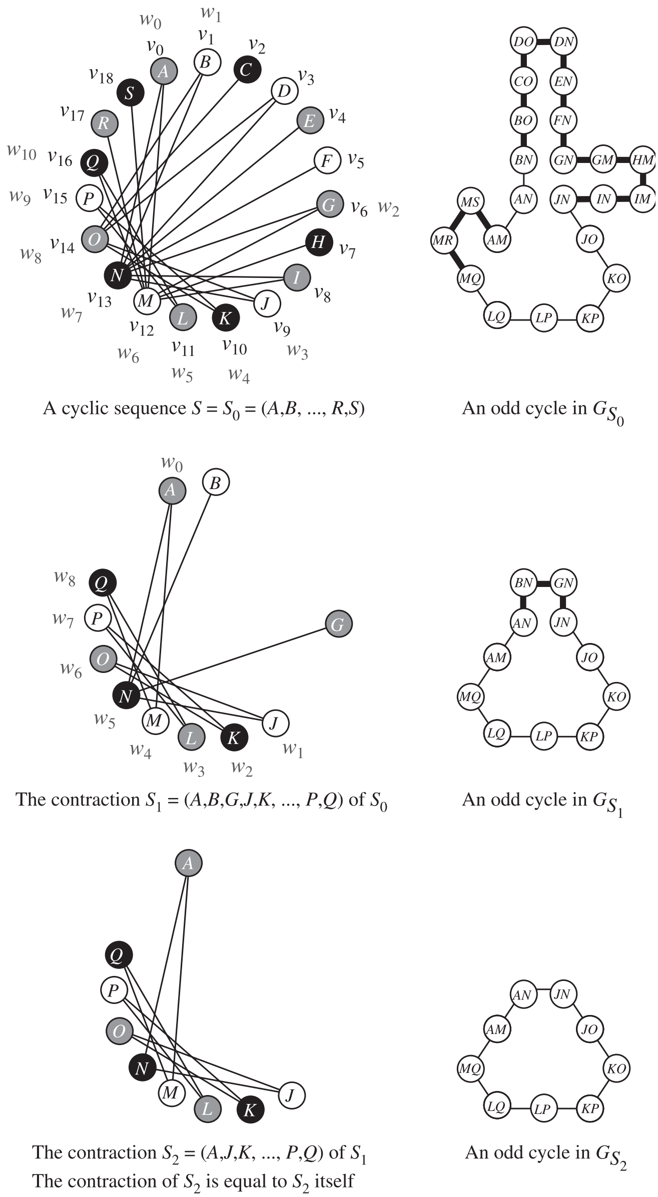


Fig. 4. Illustration of the proof of Theorem 7.

Lemma 16. *Let $\mathcal{S}'$ be the contraction of a cyclic sequence $\mathcal{S}$ with property $\mathcal{P}$. If $G_{\mathcal{S}'}$ is not bipartite, then $G_{\mathcal{S}}$ is not bipartite either.*

Proof. Assume that $G_{\mathcal{S}'}$ is not bipartite and consider two consecutive vertices (w_i, w_j) and (w_{i+1}, w_j) on an odd cycle in $G_{\mathcal{S}'}$. Let x, y and z be the three indices such that $w_i = v_x$, $w_{i+1} = v_y$, and $w_j = v_z$ on $\mathcal{S}$. Notice that v_x, v_y and v_z are pairwise non-similar. It follows from Lemma 12 that either v_{z-1} or v_{z+1} is not similar to v_x. Without loss of generality, assume $c(v_{z-1}) \neq c(v_x)$.

By construction of $\mathcal{S}'$, we have $c(v_x) = c(v_{x+2}) = \cdots = c(v_{y-1})$. Consider the sub-sequence $P = (v_x, \ldots, v_{y-1})$ of $\mathcal{S}$ and let v_{x+2h} be a vertex on P with $x + 2h \neq y - 1$. If v_z is similar to v_{x+2h+1}, then $G_{\mathcal{S}}$ contains the chain $((v_{x+2h}, v_z), (v_{x+2h}, v_{z-1}), (v_{x+2h+1}, v_{z-1}), (v_{x+2h+2}, v_{z-1}), (v_{x+2h+2}, v_z))$ having four edges. Otherwise, v_z is not similar to v_{x+2h+1}, and $G_{\mathcal{S}}$ contains the chain $((v_{x+2h}, v_z), (v_{x+2h+1}, v_z), (v_{x+2h+2}, v_z))$ with two edges. In both cases, (v_{x+2h}, v_z) is linked to (v_{x+2h+2}, v_z) in $G_{\mathcal{S}}$ by a chain having an even number of edges.

Since (v_{y-1}, v_z) is adjacent to (v_y, v_z) in $G_{\mathcal{S}}$, we have shown that $(v_x, v_z) = (w_i, w_j)$ and $(v_y, v_z) = (w_{i+1}, w_j)$ are linked in $G_{\mathcal{S}}$ by a chain having an odd number of edges. Since this is true for any two consecutive vertices on a cycle in $G_{\mathcal{S}'}$, we conclude that the existence of an odd cycle in $G_{\mathcal{S}'}$ guarantees the existence of an odd cycle in $G_{\mathcal{S}}$. □

We can now end this section with a proof of Theorem 7.

Proof of Theorem 7. Assume H is not bipartite, and let C be any induced odd chordless cycle in H. The vertices on C define a cyclic sequence $\mathcal{S} = (v_0, v_1, \ldots, v_{2k}, v_0)$. To obtain a contradiction, it follows from Lemma 11 that it is sufficient to prove that $G_{\mathcal{S}}$ is not bipartite.

Consider a series of cyclic sequences $\mathcal{S} = \mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_t$ such that $\mathcal{S}_i$ is the contraction of $\mathcal{S}_{i-1}$ $(1 \leqslant i \leqslant t)$ and the contraction of $\mathcal{S}_t$ is $\mathcal{S}_t$. It follows from Lemma 10 and from the definition of H that $\mathcal{S} = \mathcal{S}_0$ has property $\mathcal{P}$. We then know from Lemma 14 that $\mathcal{S}_i$, also has property $\mathcal{P}$ for $i = 1, \ldots, t$. Lemma 15 tells us that $G_{\mathcal{S}_t}$ is not bipartite and we know from Lemma 16 that $G_{\mathcal{S}_i}$ is not bipartite for $i = t - 1, \ldots, 0$. Hence, $G_{\mathcal{S}_0} = G_{\mathcal{S}}$ is not bipartite. □

All the concepts developed in this section are illustrated in Fig. 4. The graphs on the left-hand side correspond to the series of cyclic sequences $\mathcal{S} = \mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_t$. The graphs on the right-hand side represent an odd cycle in $G_{\mathcal{S}_i}$, $i = 0, \ldots, t$. The set of bold edges in a given $G_{\mathcal{S}_i}$ correspond to the chains of odd length which replace some edges in $G_{\mathcal{S}_{i+1}}$ (see Lemma 16). Each vertex on an odd cycle in $G_{\mathcal{S}_i}$ is represented in $\mathcal{S}_i$ by a link between the corresponding vertices.

6. The maximum stable set problem in $(S_{1,1,3}, K_{1,n})$-free graphs

In this section we prove polynomial-time solvability of the maximum stable set problem in the class of $(S_{1,1,3}, K_{1,n})$-free graphs for any fixed n. This is a consequence of the main result of this paper and the following observation.

Lemma 17. *For any integer k and n, there are finitely many connected bipartite $(S_{1,1,k}, K_{1,n})$-free graphs with a vertex of degree more than* 2.

Proof. Let G be a connected bipartite $(S_{1,1,3}, K_{1,n})$-free graph containing a vertex a_0 of degree more than 2. Denote the subset of vertices of G at distance j from a_0 by A_j. Since G is bipartite, A_j is a stable set for each j. We claim that for every $j \geqslant k + 2$, A_j is empty. Assume to the contrary that for some $j \geqslant k + 2$, A_j contains a vertex a_j, and let $a_0, a_1, \ldots, a_j$ be a shortest path connecting a_0 to a_j with $a_i \in A_i$ for $i = 0, 1, \ldots, j$. If a_2 has at least one more neighbor in A_1, say b, then the vertices $b, a_1, a_2, \ldots, a_{k+2}$ induce an $S_{1,1,k}$ in G. If a_1 is the only neighbor of a_2 in A_1, then the vertices $a_0, a_1, a_2, \ldots, a_k$ together with any two other vertices in A_1 induce an

$S_{1,1,k}$ in G. Thus, for every $j \geqslant k+2$, A_j is empty, and hence, since G is $K_{1,n}$-free, there is a constant bounding the number of vertices in A_j for $j = 0, 1, \ldots, k+1$. $\square$

Combining this lemma with the polynomial algorithm for finding augmenting chains in $S_{1,2,3}$-free graphs we conclude that

Theorem 18. *For any integer n, the maximum stable set problem in the class of* $(S_{1,1,3}, K_{1,n})$*-free graphs can be solved in polynomial time.*

Notice that if $n \geqslant 3$, then the class of $(S_{1,1,3}, K_{1,n})$-free graphs includes all claw-free graphs. Moreover, Theorem 18 generalizes polynomial-time solutions for $(P_5, K_{1,n})$-free graphs [10] and $(K_{1,2} + K_2, K_{1,n})$-free graphs [2].

7. Conclusion

We have proved that augmenting chains can be detected in polynomial time in the class of graphs without skew star. Our algorithm is very similar to Minty's algorithm for claw-free graphs. It only differs in two points: we impose an additional condition for creating an edge in the graph $H(b)$ associated with a regular black vertex b; we do not consider non-interesting IWAPs for the construction of Edmond's graph. Hence, while the proofs contained in this paper are not particularly simple, our algorithm is not more complicated than Minty's one.

As an illustration, we applied the obtained result to prove polynomial-time solvability of the maximum stable set problem in the class of $(S_{1,1,3}, K_{1,n})$-free graphs, for any fixed n, which improves several existing results. We believe also that our algorithm may lead to many other positive results for the problem in question. One possible way toward this goal is to explore the structure of other types of augmenting $S_{1,2,3}$-free graphs (which can be done on the base of the characterization of bipartite $S_{1,2,3}$-free graphs proposed in [8]) and to develop polynomial-time algorithms for detecting those graphs.

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